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Classical Frames for a Quantum Theory—A Bird's-Eye View

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All proposals of phase-space models for quantum mechanics are classified into two well-defined classes and then described in terms of operational statistical theories. The extreme generality of the description, devoid of unessential details, leads to almost trivial proofs of many facts discovered elsewhere with great effort. A new proposal of quantization is briefly sketched.

1. TWO APPROACHES

The great successes and development of quantum mechanics were accompanied by obstinate efforts to bring it back to familiar classical grounds. The never-abating motivation of these efforts comes presumably from the natural tendency to want to "understand" quantum mechanics. This means "to connect it with one's whole experience and knowledge," which are inevitably "classical." The long search for a phase-space reformulation of quantum mechanics, like the search for the philosopher's stone, resulted in a realization of the impossibility of achieving the goal. It seems clear now that the basic structures of quantum theories differ essentially from those of classical theories, hence any classical representation of quantal states *and* observables can be constructed only at the cost of destroying the original quantal theory. A quantal theory must die before it enters the classical paradise.

Nevertheless the search for phase-space models of quantal theories is not devoid of importance. The majority of physicists who are satisfied with the standard quantum mechanics consider a representation of quantal states by probability distributions over a phase space to be a convenient calculational device. For those who want to go outside the borders imposed by quantum mechanics, as well as for those interested in relations between

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different statistical theories, the classical representations should be still an object of interest.

It has appeared recently that all attempts to find satisfactory classical models for quantal theories have a common underlying structure, which makes it possible to compare and to classify them. There are two well-defined approaches. The first one we will call the state-injection approach; it comes directly from Wigner's idea of representing linearly quantum-mechanical density matrices as probability distributions over some phase space (Hillery *et al.*, 1984; O'Connell, 1988; Sudarshan, 1983). To this approach belong numerous proposals of nonnegative distribution functions representing quantal states made by Husimi, Bopp, Kano, She and Heffner, Mehta and Sudarshan, Prugovečki, and others [for references see Hillery *et al.* (1984), O'Connell (1988), Sudarshan (1983)]. The operational generalization of statistical theories in its final form (Davies and Lewis, 1970; Davies, 1976; Ludwig, 1983) made possible the discovery (Ali and Prugovečki, 1977) of a common scheme of all these proposals, which in turn led to the formulation of the elegant mathematical framework of the state-injection approach [Neumann, 1971; Busch and Lahti, 1989; Singer and Stulpe, 1991; Busch *et al.*, 1991; Bugajski *et al.*, 1993).

Proposals of the second class, which we will call the delinearization approach, consider just the quantal pure states as points of the phase space of the sought classical representation. The delinearization approach unifies at least three streams. One of them comes from critical discussions around von Neumann's description of quantum-mechanical composed systems. Here belong important papers of Misra (1974) and Ghirardi *et al.* (1976). Another one is related to attempts to formulate a geometrical framework for quantum mechanics in terms of infinite-dimensional Hamiltonian systems and to the geometric quantization program (Hermann, 1965, 1982; Kibble, 1979; Souriau, 1983; Cirelli and Lanzavecchia, 1984). It was suggested recently that the delinearization approach should also include the broad field of nonlinear generalizations of quantum mechanics (Bugajski, 1991). All formalisms in the delinearization approach must contain, which is a rather new discovery (Holevo, 1982; Neumann, 1985; Ludwig, 1990; Bugajski, 1991; Bugajski *et al.*, 1993), the same basic mathematical structure, similar to that of the state injection approach. This opens a way to formulate the fundamentals of the two approaches in a common language of statistical dualities.

2. STATISTICAL DUALITIES

The notion of statistical duality, coined by the Marburg school (Ludwig, 1983; Werner, 1983; Neumann, 1985; Stulpe, 1988), provides a natural generalization of the basic structure of operational statistical theories (Davies and Lewis, 1970; Davies, 1976; Ludwig, 1983). We define statistical duality as a pair (V, W) consisting of a base-normed Banach space V and an order-unit Banach space W together with a bilinear form $\langle\cdot,\cdot\rangle$: $V\times W\to\mathbb{R}$ placing V and W in norm and order duality. We can imagine W as a (norm-) closed subspace of the Banach dual V^*, and V as a closed subspace of W^*. The base S_V of V represents the convex set of all states of the physical statistical system described by the statistical duality. The set Ex S_V of the extreme elements of S_V represents pure states (it may be empty in general). The order interval $[o, e]_W$ of W, where e denotes the order unit and o the origin of W, represents the set of all elementary observables (effects, yes–no experiments, etc.) of the physical system in question. Its set of extreme elements Ex$[o, e]_W$ can be identified as the set of elements of the "quantum logic" of the physical system (it also may be empty in general). The real number $\langle\alpha, a\rangle$ for $\alpha\in S_V$ and $a\in[o, e]_W$ is the probability of response 1 after a single measurement of the effect a on the state α.

Among the variety of statistical dualities we should be able to point out those describing classical systems. We will assume that any classical statistical theory has to display two fundamental properties: the unique decomposability of all mixed states and the mutual compatibility (coexistence) of all observables. The first property translated into the abstract language of statistical dualities forces upon S_V (the base of V) the structure of a Bauer simplex, whereas the second makes W be a Banach lattice. It is well known (Alfsen, 1971; Asimov and Ellis, 1980; Schaefer, 1986) that under such conditions V can be identified as $M(\Omega)$, the base-normed space of all signed (Radon) measures on Ω, with $\Omega=\text{Ex } S_V$ and $S_V=M(\Omega)_1^+$ (the probability measures). The space $M(\Omega)$ is the (Banach) dual of $C(\Omega)$, it is a vector lattice, and so is its dual $M(\Omega)^*$. The characteristic functions of the Baire subsets of Ω can be identified with the extreme elements of the order interval $[o, e]_{M(\Omega)^*}$. The norm-closed subspace $F(\Omega)$ of $M(\Omega)^*$ spanned by these characteristic functions seems to fit well for the second element of the classical statistical duality. Thus we define classical duality as the pair $(M(\Omega), F(\Omega))$ of Banach spaces with $M(\Omega)$ the base-normed space of all signed Baire measures on a compact Hausdorff Ω and $F(\Omega)$ the order-unit space of (Baire) measurable real functions on Ω. The bilinear form is defined by means of an integral [for details see Singer and Stulpe (1991)].

It is not clear how to characterize quantal statistical dualities. We will simply assume that "quantal" is any statistical duality with both spaces not vector lattices.

3. THE STATE-INJECTION APPROACH

Wigner was the first to construct a linear embedding of the convex set of states of the standard quantum mechanics into $M(\Omega)$ for some phase space Ω. His mapping was not positive. For several decades this defect was considered to be inevitable on the grounds of a belief in the impossibility of representing linearly quantal states by probability measures. This belief was not impaired even by early successful realizations of such a representation. Now it is clear that a classical representation of states can be constructed for a wide class of quantal statistical dualities. The first such construction was done by Ali and Prugovečki (1977) for the states of the standard quantum mechanics.

Let (V, V^*) be a quantal statistical duality (to avoid unessential complications we assume $W = V^*$). An affine injection L: $S_V \to M(\Omega)_1^+$ for an appropriate measurable space Ω will be called (provided it does exist) a phase-space representation of S_V. As S_V generates V, L admits (if it exists) a unique linear extension $\hat{L}$: $V \to M(\Omega)$. $\hat{L}$ is positive and of norm 1, hence is norm-continuous and its dual $\hat{L}^*$: $M(\Omega)^* \to V^*$ is positive and w^*-continuous.

It is easy to realize, owing to the adopted "bird's-eye view," that L (under the permanent assumption of existence) defines a particular observable P: $\mathbb{B}(\Omega) \to V^*$ over the Boolean σ-algebra of Baire subsets of Ω. Namely $\langle \alpha, P(X) \rangle := \langle L\alpha, \chi_X \rangle = \int_\Omega \chi_X \, d(L\alpha)$, where $X \in \mathbb{B}(\Omega)$, $\alpha \in S_V$, and $\chi_X \in F(\Omega)$ is the characteristic function of X. It is evident that P is obtained simply by restricting $\hat{L}^*$ to the characteristic functions of the Baire sets.

The observable P shows a peculiar property: as any element of $M(\Omega)$ is nothing but a (signed) measure on its value space Ω, the observable P has to distinguish all of them. Hence P is able to distinguish all elements of $L(S_V)$, i.e., all quantal states: if the measure

$$\alpha \circ P: \quad \mathbb{B}(\Omega) \xrightarrow{P} [o, e]_{V^*} \xrightarrow{\alpha} [0, 1]$$

(the real unit interval) with $\alpha \in S_V$, is equal to another measure, say $\beta \circ P$, for $\beta \in S_V$, then $\alpha = \beta$. Because of this P is called an informationally complete (i.c.) observable. The first such observable was constructed by Prugovečki (1977), who also realized its close connection with the phase-space representation.

Indeed, an i.c. observable does not merely accompany a phase-space representation: given such an observable P: $\mathbb{B}(\Omega) \to [o, e]_{V^*}$ we can build up the related phase-space representation simply by defining $L(\alpha) := \alpha \circ P$ for any $\alpha \in S_V$. Any separable statistical duality (V, V^*) admits an i.c. observable, hence admits a phase-space representation [a result of Singer and Stulpe (1991)].

The brightness of the mathematical form of the state-injection approach should not hide its defects. It has to be stressed that the phase-space representation as defined above does not provide a classical representation of the quantal statistical duality. Because of its linearity $\hat{L}^*$ preserves the compatibility (coexistence) of effects; thus $\hat{L}^*([o, e]_{F(\Omega)})$ consists of mutually compatible elements. This implies that $\hat{L}^*([o, e]_{F(\Omega)})$ is always a proper subset of $[o, e]_{V^*}$, hence (assuming that $\hat{L}^*$ is bijective, which generally does not hold true) $\hat{L}^{*-1}$ attaches classical *effects* only to a narrow set of mutually compatible quantal effects; the rest are mapped outside $[o, e]_{F(\Omega)}$ and even outside the positive cone of $F(\Omega)$. A clear demonstration of this phenomenon in the special case of finite-dimensional Hilbert-space quantum theory can be found in the paper of Busch *et al.* (1991).

It is impossible to remove the above defect staying inside the state-injection approach, as the unfortunate property of $\hat{L}^*$ comes directly from the definition of the phase-space representation. So let us try to relax the assumed properties of L: we define a quasi-phase-space representation Λ of S_V (provided it does exist) as an affine injection of S_V into the hyperplane determined in $M(\Omega)$ by $M(\Omega)_1^+$. If $\Lambda(S_V)$ is a subset of $M(\Omega)_1^+$ we get the proper phase-space representation. If, conversely, $\Lambda(S_V)$ contains $M(\Omega)_1^+$, then $\hat{\Lambda}^*([o, e]_{F(\Omega)})$ contains $[o, e]_{V^*}$. Measures representing quantum states are then not positive in general; however, all quantal effects get their classical counterparts from $[o, e]_{F(\Omega)}$. We see here in its generality the effect of "shifting the burden," observed by Sudarshan (1983) in the special case of Kano distributions.

The Wigner map Λ_W lies between the two extremes: $\Lambda_W(S_V)$ is placed in such a manner that the image of any quantal pure state falls outside $M(\Omega)_1^+$ [proved by Wigner; see Hillery *et al.* (1984)]; nevertheless all extreme elements of $M(\Omega)_1^+$ lie outside $\Lambda_W(S_V)$. This immediately implies that the Wigner map suffers two defects at the same time: quantal states are represented by not necessarily nonnegative measures and quantal effects possess not necessarily positive classical counterparts.

Two obvious remarks close this section. The intermediate case $\Lambda(S_V) = M(\Omega)_1^+$ is evidently impossible because the set of quantal states must have essentially different affine structure than the set of all probability measures—the latter is a simplex, whereas the former cannot be a simplex on the strength of the definition. An interesting illustration of this fact is provided in Busch *et al.* (1991). The second remark is that it is evident that any quasi-phase-space representation (except the case when it is a proper phase-space representation), including the Wigner one, cannot possess the associated i.c. observable.

4. THE DELINEARIZATION APPROACH

One could risk the claim that the philosophical attitude toward the peculiarities of quantum mechanics in this approach is to some extent opposite to that of the state-injection approach. The representation of quantal pure states by essentially diffuse probability distributions over a phase space suggests that quantal probabilities would be caused by the imprecise preparation of states, as in the case of classical statistical theories. Behind the delinearization approach one can find a rather opposite idea: quantum pure states would be really pure, but taking too narrow a set of observables into account causes all the "genuine quantal" features of the theory.

The set of quantal pure states is then the starting place of the approach; hence we should describe a quantal system by means of a statistical duality (W^*, W) rather than (V, V^*) in order to have a sufficiently rich set $\mathrm{Ex}\, S_{W^*}$. Then we define the statistical duality $(M(\Omega), F(\Omega))$ based on $\Omega = \overline{\mathrm{Ex}\, S_{W^*}}$ (the bar denotes the w^*-closure). The linear injection D: $[o, e]_W \to [o, e]_{F(\Omega)}$ defined by the evaluation $\langle \alpha, D(a)\rangle := \langle \alpha, a\rangle$ for any $a \in [o, e]_W$ and $\alpha \in \overline{\mathrm{Ex}\, S_{W^*}}$ will be called the delinearization of the original statistical duality. D admits a unique linear extension $\hat{D}$: $W \to F(\Omega)$, $\hat{D}$ is the smallest separating functional representation of W (Alfsen, 1971), and thus it is injective and order-preserving (hence isometric). Observe that the delinearization always exists for a given duality (W^*, W) and is unique.

The map $\hat{D}$ allows us to identify W with a uniformly closed subspace of $F(\Omega)$. The adjoint map $\hat{D}^*$ can be seen now as attaching to any linear functional $\alpha \in M(\Omega)$ its restriction to W. It is essential that $\hat{D}^*$ always maps $M(\Omega)_1^*$ onto S_{W^*} and that (except for the extreme points of S_{W^*}) it attaches to a given quantal state a large collection of probability measures on Ω (Alfsen, 1971; Asimov and Ellis, 1980; Schaefer, 1986).

The delinearization approach avoids some disadvantages of the foregoing. It transforms any quantal observable A: $\mathbb{B} \to [o, e]_W$ into a well-defined and unique classical observable simply by composing it with the delinearization,

$$\hat{D} \circ A: \quad \mathbb{B} \xrightarrow{A} [o, e]_W \xrightarrow{\hat{D}} [o, e]_{F(\Omega)}$$

Evidently all quantal probability distributions

$$\alpha \circ A: \quad \mathbb{B} \xrightarrow{A} [o, e]_W \xrightarrow{\alpha} [0, 1]$$

with $\alpha \in S_{W^*}$ are preserved under delinearization: $\alpha \circ A = \beta \circ \hat{D} \circ A$ for any quantal observable A, any $\alpha \in S_{W^*}$, and any $\beta \in \hat{D}^{*-1}(\alpha)$. Nevertheless quantal observables, so carefully transported to the classical theory, become classical observables, hence all their quantal mutual relations (incompatibility, complementarity, commutation relations, etc.) disappear. Observe also that all quantal effects are mapped by D onto fuzzy classical effects (with obvious exceptions), so all original quantal observables become unsharp after delinearization.

The many-to-one property of the restriction map $\hat{D}^*$: $M(\Omega)_1^+ \to S_{W^*}$, although it can be inconvenient for concrete computations, should satisfy many theoreticians. Indeed, some efforts to introduce the quantal phase space $\Omega = \overline{\text{Ex}\, S}_{W^*}$ (Misra, 1974; Ghirardi *et al.*, 1976) were motivated by the idea of extending von Neumann's density matrix description (our S_{W^*}) into the broad set of statistical mixtures of pure states [our $M(\Omega)_1^+$].

The delinearization approach provides a new and perhaps more satisfactory phase-space framework for any quantal theory. However, it cannot be seen as a way to achieve a classical representation for quantum mechanics, because any quantal theory after delinearization becomes classical.

5. QUANTIZATION

Some of the mappings appearing in our discussion of the two approaches point from the classical toward the quantal. This suggests that these approaches could also provide methods of quantization. Indeed, the "prime quantization" of Ali and Doebner (1986) fits well the scheme of the state-injection approach: their prime quantization map (their formulas 3.10 and 3.11) appears like, save for unessential details, our map $\hat{L}^*$.

The quantization problem consists in discovering somehow a natural and convincing way to ascribe the Hilbertian quantum model to a classical theoretical model of a physical system. It is not merely a question of language, so we cannot be satisfied only by expressing a classical theoretical model in terms of operators on a Hilbert space. What is essential for the quantization is to obtain the genuine quantal relations between operators chosen to represent classical observables. Thus, if the map L^* is used to define a quantization procedure, it should be complemented by a precise prescription for how to pass from the i.c. observable and its functions to the sharp observables of standard quantum mechanics satisfying the known commutation relations.

On the other hand, the delinearization approach, once numerous concrete problems have been solved, could also provide a new method of quantization. Having a classical model of a physical system subjected to quantization, we pick up some physically basic sharp observables. Then we attach to them unsharp observables with the same expectations in all states [such constructions can be found in the literature (Davies, 1976; Busch, 1985, 1986; Ali and Prugovečki, 1977)]. The closed linear subspace of $F(\Omega)$ spanned by all effects of such chosen unsharp observables would correspond to $\hat{D}(W)$ of the delinearization approach. The restrictions of all classical states to this subspace will define the set of states of the "quantal" model we are constructing. Here the story must be interrupted—it is not known yet how to obtain a Hilbert-space representation of a statistical duality. Several other hard problems must be solved before the above scheme can become a working procedure. We can, however, illustrate the sketched quantization procedure by a simple example adopted from a paper of Neumann (1985).

Let Σ denote the unit sphere in $\mathbb{R}^3$ with the topology induced by the natural topology of $\mathbb{R}^3$. We construct the classical statistical duality, taking Σ as the phase space Ω. The duality $(M(\Sigma), F(\Sigma))$ will be considered as the classical model to be quantized. Let us now take the subspace G of $F(\Sigma)$ spanned by all functions $f(\mathbf{x}) = f_0 + \mathbf{f} \cdot \mathbf{x}$ with $(f_0, \mathbf{f}) \in \mathbb{R}^4$, $\mathbf{x} \in \Sigma$. Here G is a four-dimensional order-unit normed subspace of $F(\Sigma)$. As any element of G can be extended to an affine function on the convex set $B := \{\mathbf{x} \mid \mathbf{x} \in \mathbb{R}^3, \|\mathbf{x}\| \leq 1\}$, i.e., on the unit ball of $\mathbb{R}^3$, G can be identified with $A(B)$—the order unit Banach space of affine functions on B. Thus (Alfsen, 1971; Asimov and Ellis, 1980; Schaefer, 1986) the ball B is the base of the dual G^*, and $\Sigma = \operatorname{Ex} B$ is the set of pure states of the obtained quantal statistical duality (G^*, G). The injection $\hat{D}\colon G \to F(\Sigma)$ defines the dual map $\hat{D}^*$ which to any point of B attaches a collection of probability measures on Σ. It is rather evident now that B, considered as the set of states of (G^*, G), is to be identified with the Poincaré ball (e.g., Busch and Schroeck, 1989), so we get the two-dimensional Hilbert-space representation of the obtained quantal statistical duality. Thus our quantization procedure results in the standard description of the spin properties of a spin-half system.

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